# Spatial Reasoning with Topological Information

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**Abstract.** This chapter summarizes our ongoing research on topological spatial reasoning using the Region Connection Calculus. We are addressing different questions and problems that arise when using this calculus. This includes representational issues, e.g., how can regions be represented and what is the required dimension of the applied space. Further, it includes computational issues, e.g., how hard is it to reason with the calculus and are there efficient algorithms. Finally, we also address cognitive issues, i.e., is the calculus cognitively adequate.

## 1 Introduction

When describing a spatial configuration or when reasoning about such a configuration, often it is not possible or desirable to obtain precise, quantitative data. In these cases, qualitative reasoning about spatial configurations may be used.

Different aspects of space can be treated in a qualitative way. Among others there are approaches considering orientation, distance, shape, topology, and combinations of these. A summary of work on these and other aspects of qualitative spatial reasoning can be found in [Coh97].

One particular approach in this context has been developed by Randell, Cui, and Cohn [RCC92], the so-called *Region Connection Calculus* (RCC), which is based on binary topological relations. One variant of this calculus, RCC-8, uses eight mutually exhaustive and pairwise disjoint relations, called base relations, to describe the topological relationship between two spatial regions. A similar calculus was developed by Egenhofer [Ege91], who defined relations by comparing the intersection of the interior, the exterior, and the boundary of different planar regions and identified the same base relations.

In this chapter we are addressing different aspects of using RCC-8. Among these are cognitive aspects of RCC-8, namely, whether a formally defined topological calculus like RCC-8 can also be regarded as cognitively adequate. We will report about an empirical investigation on that topic [KRR97] that resulted from a cooperation with the project MEMOSPACE (see their chapter in this volume [KRSS98]).

One aspect is concerned with representational properties. As spatial regions used by RCC-8 are arbitrary regular subsets of the topological space, it is unclear how these regions should be represented. We will present a canonical model that allows a simple representation where regions are reduced to their important points and information about the neighborhood of these points [Ren98].

Most applications of spatial reasoning deal with two- or three-dimensional space and not with arbitrary topological space, where dimension is not considered. Therefore there might be consistent sets of RCC-8 relations which are not realizable in the desired dimension. Using the canonical model, we can prove that any consistent set is always realizable in any dimension $d \geq 1$ if arbitrary regions are used and in any dimension $d \geq 3$ if regions must be internally connected [Ren98].

Another aspect is concerned with computational issues of reasoning with RCC-8. We will prove that reasoning with RCC-8 is NP-hard in general and identify a large maximal tractable subset of RCC-8 which can be used to make reasoning much more efficient even in the general NP-hard case [RN97].

This chapter is organized as follows. In the second section we introduce RCC-8, Section 3 summarizes our empirical investigation on cognitive validity of RCC-8. In Section 4 we introduce the modal encoding of RCC-8 and identify the canonical model. In Section 5 this model will be interpreted topologically, which allows a simple representation of regions and also predications about the dimension of regions. Section 6 summarizes our results on computational properties of RCC-8.

## 2 Qualitative Spatial Reasoning with RCC

RCC is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are regular subsets of a topological space $\mathcal{U}$ [RCC92]. $\mathcal{U}$ is called the *universe*, i.e., the whole space. Relationships between spatial regions are defined in terms of the relation $C(r, s)$ which is true if and only if the closure of region $r$ is connected to the closure of region $s$, i.e. if their closures share a common point. We consider only regular closed regions, i.e., regions that are equivalent to the closure of their interior. This is no restriction, as with the above definition of $C$ it cannot be distinguished between open, semi-open, and closed regions. Regions themselves do not have to be internally connected, i.e., a region may consist of different disconnected parts. The domain of *spatial variables* (denoted as $X, Y, Z$) is the whole topological space.

In this work we will focus on RCC-8, but most of our results can easily be applied to RCC-5, a subset of RCC-8 [Ben94]. RCC-8 uses a set of eight pairwise disjoint and mutually exhaustive binary relations, called *base relations*, denoted as DC, EC, PO, EQ, TPP, NTPP, $\mathsf{TPP}^{-1}$, and $\mathsf{NTPP}^{-1}$, with the meaning *DisConnected, Externally Connected, Partial Overlap, EQual, Tangential Proper Part, Non-Tangential Proper Part*, and their converses. Examples for these relations are shown in Figure 1. In RCC-5 the boundary of a region is not taken into account, i.e., one does not distinguish between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 base relations DR for *DiscRete* and PP for *Proper Part*, respectively.

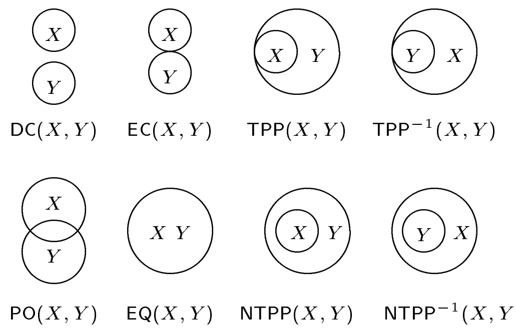


**Fig. 1.** Two-dimensional examples for the eight base relations of RCC-8

Sometimes it is not known which of the eight base relations holds between two regions, but it is possible to exclude some of them. In order to represent this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in $2^8$ different relations, including the union of all base relations, which is called *universal relation*. In the following we will write sets of base relations to denote these unions. Using this notation, the RCC-5 base relation $\mathsf{DR} = \mathsf{DC} \cup \mathsf{EC}$, e.g., is identical to $\{\mathsf{DC}, \mathsf{EC}\}$. *Spatial formulas* are written as $XRY$, where $R$ is a spatial relation. A *spatial configuration* can be described by a set $\Theta$ of spatial formulas.

Apart from union ($\cup$), other operations are defined, namely, converse ($^\smile$), intersection ($\cap$), and composition ($\circ$) of relations. The formal definitions of these operations are:

$$\begin{aligned}
\forall X, Y &: X(R \cup S)Y \leftrightarrow XRY \vee XSY, \\
\forall X, Y &: X(R \cap S)Y \leftrightarrow XRY \wedge XSY, \\
\forall X, Y &: \quad XR^\smile Y \leftrightarrow YRX, \\
\forall X, Y &: X(R \circ S)Y \leftrightarrow \exists Z : (XRZ \wedge ZSY).
\end{aligned}$$

The compositions of the eight base relations are shown in Table 1. Every entry in the composition table specifies the relation obtained by composing the base relation of the corresponding row with the base relation of the corresponding column. Composition of two arbitrary RCC-8 relations can be obtained by computing the union of the composition of the base relations.

Given a particular subset $\mathcal{S}$ of RCC-8, the *closure* of $\mathcal{S}$ under composition, intersection, and converse contains all relations that can be obtained by applying these operations to the relations of $\mathcal{S}$. The closure of $\mathcal{S}$ is denoted $\widehat{\mathcal{S}}$. The closure of the set of RCC-8 base relations $\mathcal{B}$, e.g., contains among other relations all relations in the composition table, as they can be obtained by composing the base relations.

One important computational problem is deciding *consistency* of a set $\Theta$ of spatial formulas. $\Theta$ is consistent, if it is possible to find a *realization* of $\Theta$, i.e., an instantiation of every spatial variable with a spatial region such that all relations

| ∘ | DC | EC | PO | TPP | NTPP | TPP$^{-1}$ | NTPP$^{-1}$ | EQ |
|---|---|---|---|---|---|---|---|---|
| DC | * | DC,EC PO,TPP NTPP | DC,EC PO,TPP NTPP | DC,EC PO,TPP NTPP | DC,EC PO,TPP NTPP | DC | DC | DC |
| EC | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | DC,EC PO,TPP TPP$^{-1}$,EQ | DC,EC PO,TPP NTPP | EC,PO TPP NTPP | PO TPP NTPP | DC,EC | DC | EC |
| PO | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | * | PO TPP NTPP | PO TPP NTPP | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | PO |
| TPP | DC | DC,EC | DC,EC PO,TPP NTPP | TPP NTPP | NTPP | DC,EC PO,TPP TPP$^{-1}$,EQ | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | TPP |
| NTPP | DC | DC | DC,EC PO,TPP NTPP | NTPP | NTPP | DC,EC PO,TPP NTPP | * | NTPP |
| TPP$^{-1}$ | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | EC,PO TPP$^{-1}$ NTPP$^{-1}$ | PO TPP$^{-1}$ NTPP$^{-1}$ | PO,EQ TPP TPP$^{-1}$ | PO TPP NTPP | TPP$^{-1}$ NTPP$^{-1}$ | NTPP$^{-1}$ | TPP$^{-1}$ |
| NTPP$^{-1}$ | DC,EC PO,TPP$^{-1}$ NTPP$^{-1}$ | PO TPP$^{-1}$ NTPP$^{-1}$ | PO TPP$^{-1}$ NTPP$^{-1}$ | PO TPP$^{-1}$ NTPP$^{-1}$ | PO,TPP$^{-1}$ TPP,NTPP NTPP$^{-1}$,EQ | NTPP$^{-1}$ | NTPP$^{-1}$ | NTPP$^{-1}$ |
| EQ | DC | EC | PO | TPP | NTPP | TPP$^{-1}$ | NTPP$^{-1}$ | EQ |

**Table 1.** Composition table for the eight base relations of RCC-8, where $*$ specifies the universal relation.

hold between the regions. We call this problem RSAT. For example consider the set $\Theta = \{X\{\mathsf{NTPP}\}Y, Y\{\mathsf{TPP}\}Z, Z\{\mathsf{TPP}, \mathsf{NTPP}\}X\}$. $\Theta$ is inconsistent as it follows from Table 1 that NTPP composed with TPP is NTPP, so in our example $X\{\mathsf{NTPP}\}Z$ should be true which contradicts $X\{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}Z \in \Theta$. This is easy to see, as it is not possible that a region $r$ is part of a region $s$ which is part of another region $t$ which is part of $r$. When only relations of a specific set $\mathcal{S}$ are used in $\Theta$, the corresponding reasoning problem is denoted RSAT($\mathcal{S}$).

A *canonical model* of RCC-8 is a model by which every consistent set of RCC-8 formulas can be interpreted. The standard canonical model for RCC-8 is the topological space, as every region can be interpreted as a subset of the topological space. A canonical model for Allen's interval calculus [All83], e.g., is the set of all convex intervals of real numbers. This model allows each interval to be represented using the two endpoints of the interval. Such a simple representation is not possible with the topological space as a canonical model for RCC-8.

## 3 Cognitive Plausibility of RCC-8

Qualitative temporal and spatial calculi are usually justified by application requirements and/or the introspection of the researchers developing the calculi. The cognitive significance of these calculi is usually not investigated. One exception is Allen's interval calculus, which has been analyzed from a cognitive point

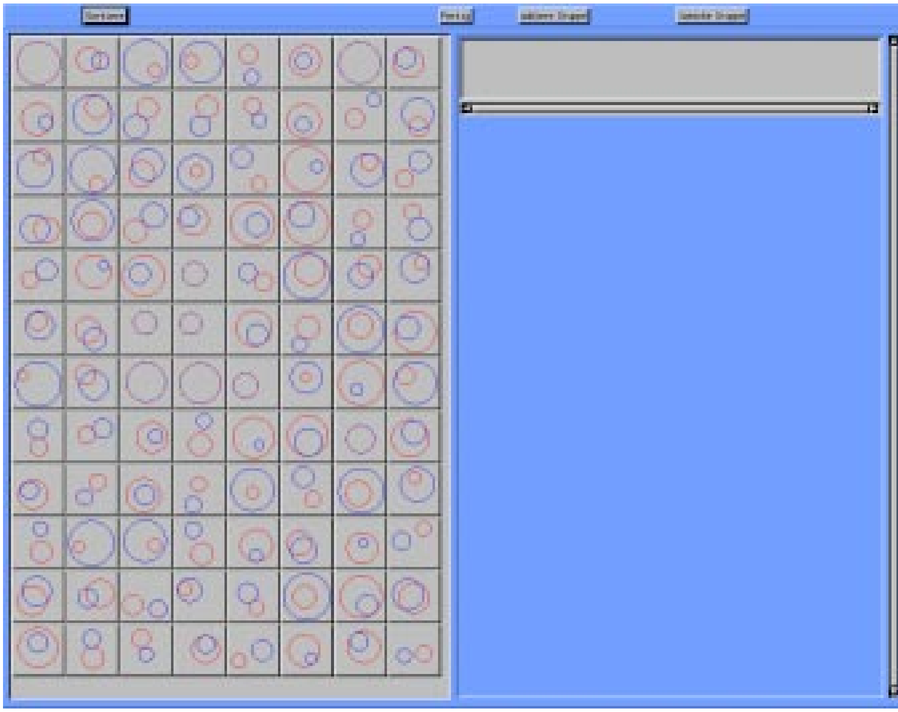

**Fig. 2.** Screen dump of the monitor at the beginning of the grouping task

of view by the MEMOSPACE project (see Chapter [KRSS98]). Here the authors distinguish between *conceptual cognitive adequacy* and *inferential cognitive adequacy* [KRS95].

According to Knauff et al [KRR97], a spatial calculus is inferentially cognitive adequate if "the reasoning mechanism of the calculus is structurally similar to the way people reason about space" and it is conceptually cognitive adequate if "empirical evidence supports the assumption that a system of relations is a model of people's conceptual knowledge of spatial relationships." Our main aim in assessing the cognitive plausibility of RCC-8 was to find out whether the distinctions made in RCC-8 are conceptually adequate. In particular, we were interested in finding out whether sub-calculi such as RCC-5 are more plausible than RCC-8. In cooperation with the MEMOSPACE project, we investigated these questions [KRR97] using the grouping task paradigm. 20 subjects (students of Albert-Ludwigs-Universität, Freiburg) were presented 96 items with varying configurations of one red and one blue circle. The task of the subjects was to group similar configurations together, where the number of groups was not given to the subjects (see Figure 2). After having completed the grouping task, subjects were (unexpectedly) asked to give natural language descriptions of the groups they had formed.

Applying a cluster analysis to the data obtained in this investigation revealed that after some clustering steps items for the RCC-8 relations were clustered together. After some more clustering steps items for the relations TPP and $\mathsf{TPP}^{-1}$ as well as items for the relations NTPP and $\mathsf{NTPP}^{-1}$ were clustered together, but at no level of the cluster analysis other sub-calculi of RCC-8 were detected. Clustering of TPP and $\mathsf{TPP}^{-1}$ as well as NTPP and $\mathsf{NTPP}^{-1}$ probably happened because some subjects ignored the distinction between reference object and to-be-localized object.

In the analysis of the natural language description of the groupings it became evident that in more than 95 % of all cases topological terms were used to describe the groupings. This and the above described finding led us to the conclusion that there is evidence that the RCC-8 system of relations is conceptually cognitive adequate, i.e., people use them to conceptualize spatial configurations [KRR97]. However, more investigations are necessary to confirm this. For instance, one should investigate whether the RCC-8 assumption of regions that are not internally connected is adequate. Further, it will be interesting to investigate the inferential cognitive adequacy of RCC-8.

## 4 Modal Encoding of RCC-8 and a Canonical Model

As RCC is defined in first-order logic, this does not lead to efficient decision procedures. It can even be derived from a result of [Grz51] that RCC is undecidable. In order to overcome this, Bennett [Ben94] used an encoding of the RCC-8 relations in propositional intuitionistic logic whereby RCC-8 is proven to be decidable. In this chapter we are using Bennett's encoding of RCC-8 in modal logic [Ben95]. After making a brief introduction to modal logic, we are describing the modal encoding and based on this identify a canonical model of RCC-8.

### 4.1 Propositional Modal Logic and Kripke Semantics

Propositional modal logic [Fit93,Che80] extends classical propositional logic by additional unary *modal operators* $\Box_i$. A common semantic interpretation of modal formulas is the *Kripke semantics* which is based on a set $W$ of so-called *worlds* and a set $\mathcal{R}$ of *accessibility relations* between these worlds, where $R \subseteq W \times W$ for every accessibility relation $R \in \mathcal{R}$. Worlds are entities in which modal formulas can be interpreted as either true or false. In different worlds modal formulas are usually interpreted differently. A different accessibility relation $R_{\Box_i}$ is assigned to every modal operator $\Box_i$. For example if $u, v \in W$ are worlds, $R_{\Box_i} \in \mathcal{R}$, and $uR_{\Box_i}v$ holds, then the world $v$ is *accessible* from $u$ with $R_{\Box_i}$. $v$ is also called $R_{\Box_i}$*-successor* of $w$.

A *Kripke model* $\mathcal{M} = \langle W, \mathcal{R}, \pi \rangle$ uses an additional valuation $\pi$ that assigns each propositional atom in each world a truth value $\{true, false\}$. Using a Kripke model, a modal formula can be interpreted with respect to the set of worlds, the accessibility relations, and the valuation. For example, a propositional atom $a$ is true in a world $w$ of the Kripke model $\mathcal{M}$ (written as $\mathcal{M}, w \Vdash a$) if and only

| *Relation* | *Model Constraints* | *Entailment Constraints* |
|---|---|---|
| $\mathsf{DC}(X,Y)$ | $\neg(X \wedge Y)$ | $\neg X, \neg Y$ |
| $\mathsf{EC}(X,Y)$ | $\neg(\mathbf{I}X \wedge \mathbf{I}Y)$ | $\neg(X \wedge Y), \neg X, \neg Y$ |
| $\mathsf{PO}(X,Y)$ | — | $\neg(\mathbf{I}X \wedge \mathbf{I}Y), X \rightarrow Y, Y \rightarrow X, \neg X, \neg Y$ |
| $\mathsf{TPP}(X,Y)$ | $X \rightarrow Y$ | $X \rightarrow \mathbf{I}Y, Y \rightarrow X, \neg X, \neg Y$ |
| $\mathsf{TPP}^{-1}(X,Y)$ | $Y \rightarrow X$ | $Y \rightarrow \mathbf{I}X, X \rightarrow Y, \neg X, \neg Y$ |
| $\mathsf{NTPP}(X,Y)$ | $X \rightarrow \mathbf{I}Y$ | $Y \rightarrow X, \neg X, \neg Y$ |
| $\mathsf{NTPP}^{-1}(X,Y)$ | $Y \rightarrow \mathbf{I}X$ | $X \rightarrow Y, \neg X, \neg Y$ |
| $\mathsf{EQ}(X,Y)$ | $X \rightarrow Y, Y \rightarrow X$ | $\neg X, \neg Y$ |

**Table 2.** Modal encoding of the eight base relations [Ben95].

if $\pi(w,a) = true$. An arbitrary modal formula is interpreted according to its inductive structure. A modal formula $\Box_i\varphi$, e.g., is true in a world $w$ of the Kripke model $\mathcal{M}$, i.e., $\mathcal{M}, w \Vdash \Box_i\varphi$, if and only if $\varphi$ is true in *all* worlds accessible from $w$ with $R_{\Box_i}$. $\mathcal{M}, w \Vdash \neg\Box_i\varphi$ if and only if there is a world accessible from $w$ with $R_{\Box_i}$ where $\varphi$ is false. The operators $\neg$, $\wedge$ and $\vee$ are interpreted in the same way as in classical propositional logic.

Different modal operators can be distinguished according to their different accessibility relations. In this chapter we are using so-called S4-operators and S5-operators. The accessibility relation of an S4-operator must be reflexive and transitive, the accessibility relation of an S5-operator must be reflexive, transitive, and euclidean. With the accessibility relation $R$ of a *strong* S5-operator all worlds are accessible from each other, i.e., $R = W \times W$. The use of Kripke models should become more clear in Section 4.3 and Section 5, where worlds and accessibility relations are displayed (see Figure 3 and Figure 4) .

## 4.2 Modal Encoding of RCC-8

The modal encoding of RCC-8 was introduced by Bennett [Ben95] and extended in [RN97]. In both cases the encoding is restricted to regular closed regions, i.e., regions which are equivalent to the closure of their interior. The modal encoding is based on a set of *model* and *entailment constraints* for each base relation, where model constraints must be true and entailment constraints must not be true. Bennett encoded these constraints in modal logic by introducing an S4-operator $\mathbf{I}$ which he interpreted as an interior operator [Ben95]. Table 2 displays these constraints for the eight base relations. Every spatial variable corresponds to a propositional atom, so the modal formula $X \wedge Y$ corresponds to the intersection of the spatial regions $X$ and $Y$, $X \vee Y$ to the union of $X$ and $Y$, $\neg X$ to the complement of $X$, and $\mathbf{I}X$ to the interior of $X$. If a modal formula $\varphi$ must be true in all worlds, then the spatial region corresponding to $\varphi$ is equal to the universe. The model constraint for the relation $\mathsf{EC}(X,Y)$, e.g., states that the complement of the intersection of the interior of region $X$ with the interior of region $Y$ is equal to the universe. This constraint guarantees that regions $X$ and $Y$ have no common interior. The entailment constraints of $\mathsf{EC}(X,Y)$ state that the complement of the intersection of region $X$ and region $Y$ is not equal to the universe. Also the complements of both $X$ and $Y$ are not equal to the universe. These constraints guarantee that regions $X$ and $Y$ have points in common and that both regions are not empty.

In order to combine the model and entailment constraints to a single modal formula, Bennett introduced a strong S5-operator $\Box$, where $\Box\varphi$ is written for every model constraint $\varphi$ and $\neg\Box\psi$ for every entailment constraint $\psi$ [Ben95]. $\Box\varphi$ can be interpreted as *the spatial region $\varphi$ is equal to the universe* and $\neg\Box\varphi$ as *the spatial region $\varphi$ is not equal to the universe.* All constraints of a single base relation are then combined conjunctively to a single modal formula. In order to represent unions of base relations, the modal formulas of the corresponding base relations are combined disjunctively. In this way every spatial formula $XRY$ can be transformed to a modal formula $m_1(XRY)$. Two additional constraints $m_2(X)$ are necessary to guarantee that only regular closed regions $X$ are used [RN97]: every region $X$ must be equivalent to the closure of its interior and the complement of a region must be an open region.[1]

$$m_2(X) = \Box(X \leftrightarrow \neg\mathbf{I}\neg\mathbf{I}X) \wedge \Box(\neg X \leftrightarrow \mathbf{I}\neg X).$$

So any set of spatial formulas $\Theta$ can be written as a single modal formula $m(\Theta)$ where $Reg(\Theta)$ is the set of spatial variables of $\Theta$:

$$m(\Theta) = \left(\bigwedge_{XRY\in\Theta} m_1(XRY)\right) \wedge \left(\bigwedge_{X\in Reg(\Theta)} m_2(X)\right).$$

As follows from the work by Bennett [Ben95], $\Theta$ is consistent if and only if $m(\Theta)$ is satisfiable.

### 4.3 A Canonical Model of RCC-8

A canonical model of a calculus is a structure that allows to model any consistent formula of the calculus. An obvious canonical model of RCC-8 is the topological space, as every spatial region can be modeled by a subset of the topological space. As described above, the modal encoding of RCC-8 can be interpreted by Kripke models. As the modal encoding of RCC-8 is equivalent to a set of RCC-8 formulas, a canonical model of RCC-8 is a structure that allows a Kripke model for any modal formula obtained by the modal encoding of RCC-8. In order to obtain a canonical model we distinguish different levels of worlds. A *world of level 0* is a world which cannot be accessed from any other world with $R_\mathbf{I}$, the accessibility relation corresponding to the $\mathbf{I}$-operator. A *world of level $l$* is a world which can be accessed with $R_\mathbf{I}$ from a world of level $l-1$ but not from other worlds with a higher level than $l-1$.

**Definition 1.** *An* RCC-8-structure $\mathcal{S}_{RCC8} = \langle W, \{R_\Box, R_\mathbf{I}\}, \pi\rangle$ *has the following properties (see Figure 3):*

[1] It can be easily verified that $\neg\mathbf{I}\neg\varphi$ corresponds to the closure of $\varphi$.

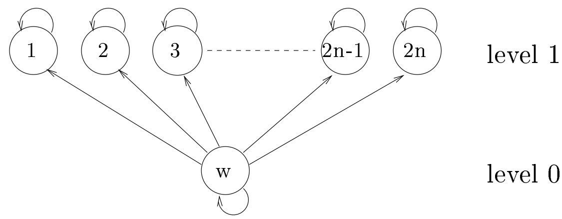


**Fig. 3.** A world $w$ of level 0 together with its $2n$ $R_{\mathbf{I}}$-successors as used in an RCC-8-structure. Worlds are drawn as circles, the arrows indicate the accessibility of worlds with the relation $R_{\mathbf{I}}$

1. *There are only worlds of level* 0 *and* 1.
2. *For every world* $u$ *of level* 0 *there are exactly* $2n$ *worlds* $v$ *of level* 1 *with* $uR_{\mathbf{I}}v$.
3. *For every world* $u$ *of level* 1 *there is exactly one world* $w$ *of level* 0 *with* $wR_{\mathbf{I}}u$.
4. *For all worlds* $w, v \in W$: $wR_{\mathbf{I}}w$ *and* $wR_{\Box}v$.

$\mathcal{S}_{RCC8}$ *contains worlds with all possible instantiations with respect to* $R_{\Box}$ *and* $R_{\mathbf{I}}$. *An* RCC-8-*model* $\mathcal{M}$ *of* $m(\Theta)$ *is a finite subset of* $\mathcal{S}_{RCC8}$. *In a polynomial* RCC-8-*model the number of worlds is polynomially bounded by the number of regions.*

Every world of level 0 together with its $2n$ $R_{\mathbf{I}}$-successors forms an independent cluster (see Figure 3). From the definition of "level" and Definition 1 it follows that $R_{\mathbf{I}}$ is reflexive and transitive, so it is guaranteed that $\mathbf{I}$ is an S4-operator. As the number of regions is countable, the number of worlds of $W$ is also countable.

**Lemma 1.** *If* $m(\Theta)$ *is satisfiable, then there is a polynomial* RCC-8-*model* $\mathcal{M}$ *with* $\mathcal{M}, w \Vdash m(\Theta)$ *with at most* $3n^2$ *worlds of level* 0.

Therefore the RCC-8-structure is a canonical model of the modal encoding of any set of spatial formulas. The number of required worlds of level 0 results from the number of different entailment constraints.

## 5 Representational Properties of RCC-8

It was shown in the previous section that the RCC-8-structure is a canonical model of RCC-8. This model was obtained from the modal encoding of topological relations, so the model depends mainly on the modal encoding but not on topology. In order to use this model for representational purposes, we have to find a way to interpret it topologically. Then the model can also be used for dealing with other properties of regions, e.g., dimension. A more detailed description of representational issues of RCC-8 can be found in [Ren98].

### 5.1 Topological Interpretation of the RCC-8 Model

The modal encoding of RCC-8 was obtained by introducing a modal operator $\mathbf{I}$ corresponding to the topological interior operator and transferring the topological properties and axioms to modal logic. Using the intended interpretation of $\mathbf{I}$ as an interior operator, it is unclear how the RCC-8-model, especially the accessibility relations $R_\Box$ and $R_\mathbf{I}$, can be topologically interpreted. In this section we present a way of topologically interpreting the RCC-8-model such that all parts of the model can be interpreted consistently. The $\mathbf{I}$-operator will not be interpreted as an interior operator, but we will prove that it satisfies the intended interpretation of an interior operator.

Because $\mathbf{I}$ is an S4-operator and because of the additional constraints $m_2$, exactly one of the following formulas is true for every world $w$ of $\mathcal{M}$ and every region $X$.

1. $\mathcal{M}, w \Vdash \mathbf{I}X$
2. $\mathcal{M}, w \Vdash \mathbf{I}\neg X$
3. $\mathcal{M}, w \Vdash X \wedge \neg\mathbf{I}X$

Consider a particular world $w$. Then the set of all spatial variables can be divided into three disjoint sets according to which of the three possible formulas is true in $w$. Let $\mathcal{X}_w$ be the set of spatial variables where the first formula is true in $w$, $\mathcal{Y}_w$ be the set where the second formula is true in $w$, and $\mathcal{Z}_w$ be the set where the third formula is true in $w$, i.e., $\mathcal{M}, w \Vdash \mathbf{I}X_i \wedge \mathbf{I}\neg Y_j \wedge (Z_k \wedge \neg\mathbf{I}Z_k)$ for all $X_i \in \mathcal{X}_w$, $Y_j \in \mathcal{Y}_w$, and $Z_k \in \mathcal{Z}_w$ .

Some relations between these spatial variables cannot hold as they contradict the modal and entailment constraints of these relations. In the following table the excluded relations and their topological consequences are shown for two regions $X$ and $Y$. $i(.)$ denotes the interior, $e(.)$ the exterior, and $b(.)$ the boundary of a region.

| Set of $X$ | Set of $Y$ | Impossible relations | Consequences |
|---|---|---|---|
| $\mathcal{X}_w$ | $\mathcal{X}_w$ | DC, EC | $i(X) \cap i(Y) \neq \emptyset$ |
| $\mathcal{X}_w$ | $\mathcal{Y}_w$ | TPP, NTPP, EQ | $i(X) \cap e(Y) \neq \emptyset$ |
| $\mathcal{X}_w$ | $\mathcal{Z}_w$ | DC, EC, TPP, NTPP, EQ | $i(X) \cap b(Y) \neq \emptyset$ |
| $\mathcal{Y}_w$ | $\mathcal{Y}_w$ | – | – |
| $\mathcal{Y}_w$ | $\mathcal{Z}_w$ | $\mathsf{TPP}^{-1}$, $\mathsf{NTPP}^{-1}$, EQ | $e(X) \cap b(Y) \neq \emptyset$ |
| $\mathcal{Z}_w$ | $\mathcal{Z}_w$ | DC, NTPP, $\mathsf{NTPP}^{-1}$ | $b(X) \cap b(Y) \neq \emptyset$[2] |

It can be seen, e.g., that when $\mathbf{I}X$ and $\mathbf{I}Y$ is true for a world $w$ then the two regions $X$ and $Y$ have a common interior.

Considering points in the topological space, we can distinguish three different ways how a point $p$ can be related to a region $X$:

[2] Actually this is not necessarily the case for $\mathsf{PO}(X, Y)$ if $X$ or $Y$ are not internally connected, but assuming this does not contradict any constraint since RCC-8 is not expressive enough to distinguish different kinds of partial overlap.

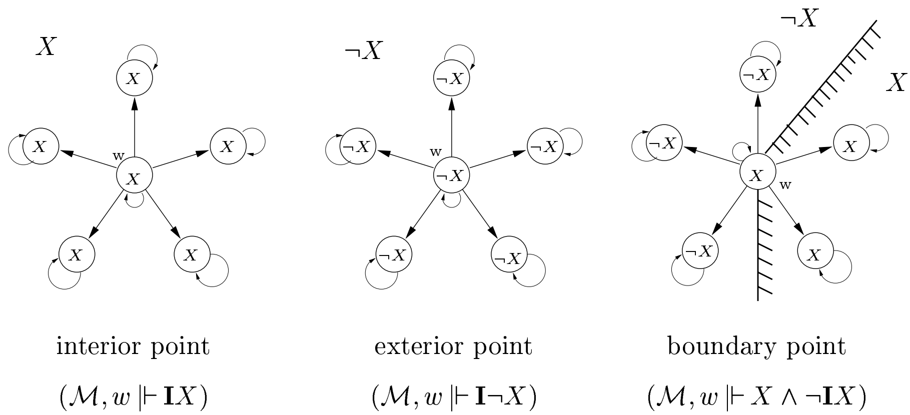


**Fig. 4.** Three different topological interpretations of a world $w$. The solid line is the boundary of $X$ where the hatched region indicates the interior of $X$.

1. $p$ interior point of $X$: there is a neighborhood $N$ of $p$ such that all points of $N$ are contained in $X$
2. $p$ is exterior point of $X$: there is a neighborhood $N$ of $p$ such that no point of $N$ is contained in $X$
3. $p$ is boundary point of $X$: every neighborhood $N$ of $p$ contains points inside of $X$ and points outside of $X$

Comparing this to the three modal formulas described above, it can be seen that there is a connection between the modal formula which is true in a world $w$ and the topological properties of a point $p$. It can be proven that there are functions $p : W \mapsto \mathcal{U}$ and $N : W \mapsto 2^{\mathcal{U}}$ that map every world $w$ to a point $p(w)$ in the topological space and to a neighborhood $N(w)$ of $p(w)$ such that

$$\begin{aligned} p(w) \in X &\text{ if } \pi(w, X) = \text{true}, \\ p(w) \notin X &\text{ if } \pi(w, X) = \text{false}, \\ p(u) \in N(w) &\text{ if } wR_{\mathbf{I}}u. \end{aligned}$$

For this proof we assume that $p(w)$ is in the interior of all regions $X_i$, in the exterior of all regions $Y_j$, and on the boundary of all regions $Z_k$ simultaneously. As there is no contradiction to this neither from the topological constraints nor from the modal constraints, it can be safely assumed. With this assumption the proof is immediate. Figure 4 shows the three different kinds of interpretations of worlds as points.

Modal formulas can now be transformed stepwise to topological formulas as follows:

$$\mathcal{M}, w \Vdash \Box\varphi \mapsto \forall u : p(u) \in \mathcal{U}. \mathcal{M}, u \Vdash \varphi$$

$$\mathcal{M}, w \not\Vdash \Box\varphi \mapsto \exists u : p(u) \in \mathcal{U}.\mathcal{M}, u \not\Vdash \varphi$$
$$\mathcal{M}, w \Vdash \mathbf{I}\varphi \mapsto \forall u : p(u) \in N(w).\mathcal{M}, u \Vdash \varphi$$
$$\mathcal{M}, w \not\Vdash \mathbf{I}\varphi \mapsto \exists u : p(u) \in N(w).\mathcal{M}, u \not\Vdash \varphi$$
$$\mathcal{M}, w \Vdash X \mapsto p(w) \in X$$
$$\mathcal{M}, w \not\Vdash X \mapsto p(w) \notin X$$

Therefore $\mathcal{M}, w \Vdash \mathbf{I}X$ can be interpreted as "there is a neighborhood $N(w)$ of $p(w)$ such that all points of $N(w)$ are in $X$". This satisfies the intended interpretation of $\mathbf{I}$ as an interior operator, as $\mathcal{M}, w \Vdash X$ means that $p(w)$ is in $X$ and $\mathcal{M}, w \Vdash \mathbf{I}X$ means that $p(w)$ is in the interior of $X$.

### 5.2 Dimension of Spatial Regions

The topological space we have been using so far does not have any particular dimension. This means that a consistent set of spatial relations is realizable in some dimension, but not necessarily in the dimension an application requires, e.g., two- or three-dimensional space. In the following we examine what dimension a space requires in order to realize the canonical model. Suppose that all $R_{\mathbf{I}}$ successors of a world $w$ are mapped to points on the boundary of an n-dimensional sphere with $p(w)$ in the center. Then the neighborhoods of Figure 4, e.g., can as shown in the figure be mapped to a two-dimensional plane where all regions are also two-dimensional. This is possible because the mappings of the $R_{\mathbf{I}}$-successors of the rightmost level 0 world can be separated by two line-segments belonging to the boundary of $X$. If the worlds cannot be separated by two line-segments for a region, we have to find a permutation of the $R_{\mathbf{I}}$-successors such that a separation is possible. A separation is necessary only for those neighborhoods that contain boundary points of a region, as for the other neighborhoods all points are the same. By analyzing which points are boundary points of which regions and the relationship between those regions, it turns out that a permutation can always be found such that the worlds can be separated by at most two line-segments for any region. In fact only two distinct $R_{\mathbf{I}}$-successors are necessary for each world of level 0. Therefore we obtain another canonical model for RCC-8 which allows models which are much more compact than the RCC-8-model as introduced in the previous section. The new canonical model is denoted *reduced* RCC-8-*structure* and the corresponding Kripke models *reduced* RCC-8-*models*. One world of level 0 of the reduced RCC-8-structure together with its $R_{\mathbf{I}}$-successors is shown in Figure 5a.

In order to obtain regions from the neighborhoods we have to *close* every neighborhood, i.e., for every neighborhood $N(w)$ find the closure of the part of every region which is affiliated with $N(w)$. Both sides of every neighborhood (see Figure 6a) can be treated almost independently. All regions which are affiliated with the same side of a neighborhood are either overlapping or one is part of the other, i.e., TPP or NTPP. For the closure of the neighborhoods all "part of" relations must be fulfilled, the partial overlap relation is not important.

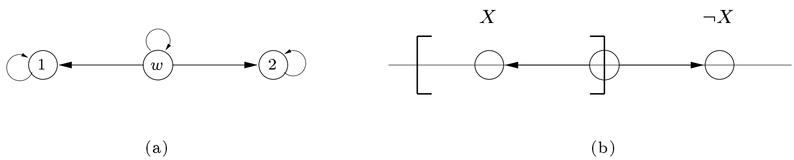


**Fig. 5.** (a) shows a world $w$ of level 0 of the reduced RCC-8-structure together with its two $R_{\mathbf{I}}$-successors. In (b) it is shown how the neighborhoods can be placed in one-dimensional space. The two brackets indicate a possible one-dimensional region $X$ where the neighborhood defines a boundary point of $X$.

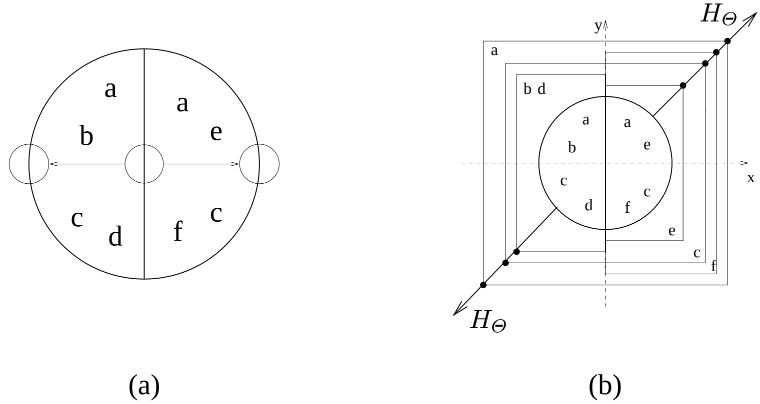


**Fig. 6.** (a) shows the two-dimensional neighborhood of a boundary point which is divided by the boundary. In (b) the neighborhood is closed with respect to the hierarchy $H_\Theta$ of the affiliated regions.

In order to fulfill the "part of" relations, we have to find a *hierarchy* $H_\Theta$ of the regions such that such that $a$ "part of" $b$ if and only if $H_\Theta(a) < H_\Theta(b)$. The parts of all regions affiliated with a neighborhood can then be closed as rectangles according to the hierarchy $H_\Theta$, i.e., regions of the same level are equal (for a particular neighborhood) and are part of all regions of a higher level (see Figure 6b). A neighborhood can be closed in any higher dimension $d$. The hierarchy of regions is then measured along the diagonal of the $d$-dimensional hypercube. In Figure 5b it can be seen that using the reduced RCC-8-model it is also possible to place the neighborhood in a one-dimensional space where regions are disconnected intervals.

**Theorem 1.** *If a set of spatial formulas $\Theta$ is consistent, the* RCC-8-*model can be realized in any dimension $d \geq 1$.*

Starting from a two-dimensional model of possibly non connected regions, it is possible to construct a three-dimensional model of connected regions.

**Theorem 2.** *If a set of spatial formulas $\Theta$ is consistent, the* RCC-8*-model can be realized in any dimension $d \geq 3$ using only connected regions.*

The new canonical model is much better suited for representational purposes than the RCC-8-model, but, as we will see in Section 6, it has some computational drawbacks.

### 5.3 Representing Regions with the Canonical Model

The RCC-8-models give us a possibility to represent topological regions. With the topological interpretation of the model it becomes clear that regions can be reduced to points and information about their neighborhood. The points that are needed within the model represent the important features of the regions with respect to a set of relations.

Using the canonical model we can give algorithms to generate a realization of $\Theta$ in the desired dimension. This can be done by simply placing the level 0 worlds together with their neighborhoods in the desired space and close the neighborhoods according to the hierarchy $H_{\Theta}$. In this realization every region consists of many disconnected parts (at most $3n^2$ pieces, as there are at most that many distinct worlds of level 0, i.e., neighborhoods). A realization using only internally connected regions can be generated in any dimension $d \geq 3$ by connecting all parts of a region of the $d-1$ dimensional realization in a specific way [Ren98].

## 6 Computational Properties of RCC-8

In order to get a deeper insight into a problem and to find efficient algorithms, an analysis of the computational properties is helpful. First results on computational properties of RCC-8 were obtained by Nebel, who considered sets of base relations [Neb95]. It was shown that the consistency problem RSAT($\mathcal{B}$) (where $\mathcal{B}$ is the set of RCC-8 base relations) is polynomial and that the path-consistency method (see also Section 6.3), a popular $O(n^3)$ approximation algorithm, is sufficient for deciding consistency. Based on these results we are interested in the complexity of the general consistency problem of RCC-8, where all 256 relations are allowed. In this section we will show that RSAT is NP-hard, i.e., that every algorithm is expected to take time super-polynomial in the number of spatial regions, provided P $\neq$ NP. As we now have intractability of the general consistency problem of RCC-8 and tractability of a subset of RCC-8, we are interested in the boundary between tractability and intractability. Therefore we identify a maximal tractable subset of RCC-8 and prove that the path-consistency method is sufficient for deciding consistency of this set. A more detailed description of the computational properties of RCC-8 can be found in [RN97]

### 6.1 Complexity of RCC-8

All of the following NP-hardness proofs use a reduction of a propositional satisfiability problem to RSAT($\mathcal{S}$) by constructing a set of spatial formulas $\Theta$ for every instance $\mathcal{I}$ of some propositional problem, such that $\Theta$ is consistent if and only if $\mathcal{I}$ is a positive instance. These satisfiability problems include 3SAT where all clauses have exactly 3 literals, NOT-ALL-EQUAL-3SAT where every clause has at least one true and one false literal, and ONE-IN-THREE-3SAT where exactly one literal in every clause must be true [GJ79].

The reductions have in common that every literal as well as every literal occurrence $L$ is reduced to two spatial variables $X_L$ and $Y_L$ and a relation $R = R_t \cup R_f$, where $R_t \cap R_f = \emptyset$ and $X_L R Y_L$ holds. $L$ is true if and only if $X_L R_t Y_L$ holds and false if and only if $X_L R_f Y_L$ holds. Additional "polarity" constraints have to be introduced to assure that for the spatial variables $X_{\neg L}$ and $Y_{\neg L}$, corresponding to the negation of $L$, $X_{\neg L} R_t Y_{\neg L}$ holds if and only if $X_L R_f Y_L$ holds, and *vice versa*. Using these polarity constraints, spatial variables of negative literal occurrences are connected to the spatial variables of the corresponding positive literal, and likewise for positive literal occurrences and negative literals. Further, "clause" constraints have to be added to assure that the clause requirements of the specific propositional problem are satisfied in the reduction. We will first prove that the consistency problem for RCC-5 is NP-hard.

**Theorem 3.** RSAT(RCC-5) *is* NP*-hard.*

**Proof Sketch.** Transformation of NOT-ALL-EQUAL-3SAT to RSAT(RCC-5) (see also [GPP95]). $R_t = \{\mathsf{PP}\}$ and $R_f = \{\mathsf{PP}^{-1}\}$. Polarity constraints:

$$X_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{\neg L}, Y_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_L\{\mathsf{PO}\}Y_{\neg L}, Y_L\{\mathsf{PO}\}X_{\neg L}.$$

Clause constraints for every clause $c = \{i, j, k\}$:

$$X_i\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_j, X_j\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_k, X_k\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_i,$$
$$X_i\{\mathsf{PO}\}Y_k, X_j\{\mathsf{PO}\}Y_i, X_k\{\mathsf{PO}\}Y_j.$$

■

Since RCC-5 is a subset of RCC-8, this result can be easily applied to RCC-8.

**Corollary 1.** RSAT(RCC-8) *is* NP*-hard.*

In the above NP-hardness proof only the relations $\{\mathsf{PO}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$, and the universal relation were used, so this set of three relations is already NP-hard. The same or similar proofs can be carried out when we use one of the RCC-8 relations $\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\}$, $\{\mathsf{TPP}, \mathsf{TPP}^{-1}\}$, $\{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$, $\{\mathsf{NTPP}, \mathsf{TPP}^{-1}\}$ or $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$ instead of $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$, so these sets are also NP-hard. The number of intractable subsets can be increased by using an additional property [NB95].

**Theorem 4.** RSAT($\widehat{\mathcal{S}}$) *can be polynomially reduced to* RSAT($\mathcal{S}$)

**Corollary 2.** *Let* $\mathcal{S}$ *be a subset of* RCC-8*.*

1. RSAT($\widehat{\mathcal{S}}$) $\in$ P *if and only if* RSAT($\mathcal{S}$) $\in$ P*.*
2. RSAT($\mathcal{S}$) *is* NP*-hard if and only if* RSAT($\widehat{\mathcal{S}}$) *is* NP*-hard.*

With this property, all sets of RCC-8 relations whose closure contains one of the five relations mentioned above are also intractable. By computing the closure of all sets containing all base relations plus one additional relation, it turned out that for 72 relations deciding consistency is NP-hard when one of them is added to the base relations.

**Lemma 2.** RSAT($\mathcal{S}$) *is* NP*-hard for any subset* $\mathcal{S}$ *of* RCC-8 *containing all base relations together with one of the 72 relations of the following sets:*

$$\mathcal{N}_1 = \{R \,|\, \{\mathsf{PO}\} \not\subseteq R \text{ and } (\{\mathsf{TPP}, \mathsf{TPP}^{-1}\} \subseteq R \text{ or } \{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\} \subseteq R)\},$$
$$\mathcal{N}_2 = \{R \,|\, \{\mathsf{PO}\} \not\subseteq R \text{ and } (\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\} \subseteq R \text{ or } \{\mathsf{TPP}^{-1}, \mathsf{NTPP}\} \subseteq R)\}.$$

### 6.2 Tractable Subsets

In order to identify a set of RCC-8 relations as tractable, one either has to specify a particular algorithm for deciding consistency of this set, or find another tractable decision problem to which the consistency problem of the particular set can be reduced. We have chosen HORNSAT, the tractable satisfiability problem of propositional Horn formulas, i.e., those propositional formulas where each clause contains at most one positive literal. For this reduction we first reduce RSAT to SAT, the propositional satisfiability problem, and then identify the relations which are reduced to Horn formulas.

For reducing RSAT to SAT, we specify a transformation by which every instance of RSAT, i.e., every set of spatial formulas $\Theta$, is transformed to a propositional formula. For this we will start from the modal encoding $m(\Theta)$ and the corresponding RCC-8-model $\mathcal{M}$. Every world $w$ of level 0 of $\mathcal{M}$ together with every spatial region $X$ results in a propositional atom $X_w$. In order to preserve the structure of the RCC-8-model in the propositional formula, the $2n$ worlds of level 1 of every level 0 world $w$ are transformed to propositional atoms $X_w^i$ for $i = 1,..,2n$. Using these atoms, every model and every entailment constraint can be transformed to a propositional formula. Additionally, the properties of the **I**-operator, i.e., reflexivity and transitivity and the $m_2$-formulas, also have to be transformed to a propositional formula. It turns out that all these formulas can be written as Horn formulas. As some of the model constraints can be transformed to indefinite Horn formulas, i.e., formulas where all clauses contain only negative literals, disjunctions of these constraints with any other constraint can also be transformed to Horn formulas. Thus every relation that can be written as a conjunction of constraints and Horn transformable disjunctions of constraints can be transformed to a Horn formula. For the set of these relations deciding consistency is thereby tractable. This set consists of 64 different relations and is denoted $\mathcal{H}_8$. Because of Corollary 2, the closure $\widehat{\mathcal{H}}_8$ of $\mathcal{H}_8$ is also tractable.

**Lemma 3.** RSAT($\widehat{\mathcal{H}}_8$) *can be polynomially reduced to* HORNSAT*.*

The reduction to HORNSAT is not possible for the reduced RCC-8-model, as the transformation of the first part of $m_2$ does not result in a Horn formula.

**Theorem 5.** $\widehat{\mathcal{H}}_8$ *contains the following* 148 *relations:*

$$\widehat{\mathcal{H}}_8 = \mathsf{RCC\text{-}8} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$$

*with* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *as defined in Lemma 2 and*

$$\mathcal{N}_3 = \{R | \{\mathsf{EQ}\} \subseteq R \text{ and } ((\{\mathsf{NTPP}\} \subseteq R, \{\mathsf{TPP}\} \not\subseteq R) \\ \text{or } (\{\mathsf{NTPP}^{-1}\} \subseteq R, \{\mathsf{TPP}^{-1}\} \not\subseteq R))\}.$$

For proving that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8, we have to show that no relation of $\mathcal{N}_3$ can be added to $\widehat{\mathcal{H}}_8$ without making RSAT intractable. For relations of the sets $\mathcal{N}_1$ and $\mathcal{N}_2$ this is already known (see Lemma 2). The following Lemma can be proven by a computer assisted case-analysis.

**Lemma 4.** *The closure of every set containing* $\widehat{\mathcal{H}}_8$ *and one relation of* $\mathcal{N}_3$ *contains the relation* $\{\mathsf{EQ}, \mathsf{NTPP}\}$.

Therefore it is sufficient to prove NP-hardness of $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$ for showing that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

**Lemma 5.** $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$ *is* NP*-hard.*

**Proof Sketch.** Transformation of 3SAT to $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$. $R_t = \{\mathsf{NTPP}\}$ and $R_f = \{\mathsf{EQ}\}$. Polarity constraints:

$$X_L\{\mathsf{EC}, \mathsf{NTPP}\}X_{\neg L}, Y_L\{\mathsf{TPP}\}Y_{\neg L}, \\ X_L\{\mathsf{TPP}, \mathsf{NTPP}\}Y_{\neg L}, Y_L\{\mathsf{EC}, \mathsf{TPP}\}X_{\neg L},$$

Clause constraints for each clause $c = \{i, j, k\}$:

$$Y_i\{\mathsf{NTPP}^{-1}\}X_j, Y_j\{\mathsf{NTPP}^{-1}\}X_k, Y_k\{\mathsf{NTPP}^{-1}\}X_i.$$

■

**Theorem 6.** $\widehat{\mathcal{H}}_8$ *is a maximal tractable subset of* RCC-8*.*

It has to be noted that there might be other maximal tractable subsets of RCC-8 that contain all base relations, since, e.g., $\mathsf{RSAT}(\{\mathsf{EQ}, \mathsf{NTPP}\} \cup \mathcal{B})$ has not been shown to be NP-hard so far.

### 6.3 Applicability of Path-Consistency

The path-consistency method is a very popular approximation algorithm for deciding consistency of a Constraint Satisfaction Problem (CSP). It can be applied since RSAT is a CSP where variables are nodes and relations are arcs of the constraint graph and the domain of the variables is the topological space. The path-consistency method imposes path-consistency of a CSP by successively removing relations from all edges with the following operation until a fixed point is reached:

$$\forall k : R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$$

where $i, j, k$ are nodes and $R_{ij}$ is the relation between $i$ and $j$. The resulting CSP is equivalent to the original CSP with respect to consistency. If the empty relation occurs while performing this operation, the CSP is inconsistent, otherwise the resulting CSP is path-consistent. More advanced algorithms impose path-consistency in time $O(n^3)$ where $n$ is the total number of nodes in the graph [MF85].

It has already been mentioned that the path-consistency method is sufficient for deciding consistency of sets of base relations. It can be shown that it is also sufficient for deciding consistency of sets of $\widehat{\mathcal{H}}_8$ relations. This is done by showing that the path-consistency method finds an inconsistency whenever positive unit resolution (PUR) resolves the empty clause from the corresponding propositional formula. The only way to get the empty clause is resolving a positive and a negative unit clause of the same variable. Since the Horn formulas that are used contain only a few types of different clauses, there are only a few ways to resolve unit clauses using PUR which were covered by a case-analysis. As PUR is refutation-complete for Horn formulas [HW74], it follows that the path-consistency method decides $\mathsf{RSAT}(\mathcal{H}_8)$. Using the proof of Theorem 4, it is possible to express every relation of $\widehat{\mathcal{H}}_8$ as a Horn formula. Then the following theorem can be proven.

**Theorem 7.** *The path-consistency method decides* $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$.

### 6.4 Applicability of the Maximal Tractable Subset

One obvious advantage of the maximal tractable subset $\widehat{\mathcal{H}}_8$ is that the path-consistency method can now be used to decide RSAT when only relations of $\widehat{\mathcal{H}}_8$ are used and not only when base relations are used.

As in the case of temporal reasoning, where the usage of the maximal tractable subset ORD-HORN has been extensively studied [Neb97], $\widehat{\mathcal{H}}_8$ can also be used to speed up backtracking algorithms for the general NP-complete RSAT problem. Previously, every spatial formula had to be refined to a base relation before the path-consistency method could be applied to decide consistency. In the worst case this has to be done for all possible refinements. Supposing that the relations are uniformly distributed, the average branching factor, i.e. the average number of different refinements of a single relation to relations of $\mathcal{B}$ is 4.0.

Using our results it is sufficient to make refinements of all relations to relations of $\widehat{\mathcal{H}}_8$. Except for four relations, every relation not contained in $\widehat{\mathcal{H}}_8$ can be expressed as a union of two relations of $\widehat{\mathcal{H}}_8$, the four relations can only be expressed as a union of three $\widehat{\mathcal{H}}_8$ relations. This reduces the average branching factor to 1.4375. Both branching factors are of course worst-case measures because the search space can be considerably reduced when path-consistency is used as a forward checking method [LR97].

The following table shows the worst-case running time for the average branching factors given above. All running times are computed as $b^{(n^2-n)/2}$ where $b$ is the average branching factor and $n$ the number of spatial variables contained in $\Theta$. We assumed that 100.000 path-consistency checks can be performed per second.

| #spatial variables | $\mathcal{B}$ (4.0) | $\widehat{\mathcal{B}}$ (2.5) | $\widehat{\mathcal{H}}_8$ (1.4375) |
|---|---|---|---|
| 5 | 10sec | 95msec | 3msec |
| 7 | 500days | 38min | 20msec |
| 10 | $10^{14}$years | $10^{6}$years | 2min |

Recent experiments have shown that consistency can be decided much faster than these numbers indicate. Almost all instances up to a problem size of 100 spatial variables can be solved in less than a second. Using $\widehat{\mathcal{H}}_8$ for the backtracking search turns out to be about twice as fast in average than using $\widehat{\mathcal{B}}$. Also a significantly larger number of difficult instances can be solved in reasonable time when $\widehat{\mathcal{H}}_8$ is used.

## 7 Summary

In this chapter we reported about our ongoing work on the cognitive, representational, and computational aspects of the Region Connection Calculus. We made an empirical investigation of whether or not people use similar topological information as in RCC-8 when conceptualizing spatial arrangements and found that RCC-8 is a good candidate for a cognitively adequate spatial relation system and that RCC-5 and other sub calculi of RCC-8 are not cognitively adequate. We introduced a new canonical model of RCC-8 that resulted from the encoding of RCC-8 in modal logic. This model was topologically interpreted which allows a more simple representation of regions than it is possible with the topological space as a canonical model. It could also be shown that a consistent set of relations always has a realization in any dimension $d \geq 3$ when regions are internally connected and $d \geq 1$ otherwise. The consistency problem of RCC-8 was shown to be intractable in general, but a maximal tractable subset of RCC-8 was identified. For this set the path-consistency method was proven to be sufficient for deciding consistency.

Open problems and further work on the topics of this chapter includes a more detailed analysis of the canonical model with respect to models of internally connected two-dimensional regions. Another open problem is whether the maximal tractable subclass we found is the only one containing all base relations. We are planning to make further empirical investigations on the cognitive validity of RCC-8. This includes studying the inferential cognitive adequacy of RCC-8 as well as examining whether the complexity results have any cognitive meaning.

**Acknowledgments**

We would like to thank Markus Knauff and Reinhold Rauh for their collaboration concerning the investigation of the cognitive aspects of RCC-8, Ronny Fehling and Thilo Weigel for their assistance in developing the software, and Markus Knauff and Fritz Wysotzki for their helpful comments on earlier versions of this chapter.

This research has been supported by DFG as part of the project FAST-QUAL-SPACE, which is part of the DFG special research effort on "Spatial Cognition."

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